

ON BRAIDED QUANTUM GROUPS

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ABSTRACT. A braided generalization of the concept of Hopf algebra (quantum group) is presented. The generalization overcomes an inherent geometrical inhomogeneity of quantum groups, in the sense of allowing completely pointless objects. All braid-type equations appear as a consequence of initial axioms. Braided counterparts of basic algebraic relations between fundamental entities of the standard theory are found.

1. INTRODUCTION

The aim of this study is to present abstract elements of a braided theory which generalizes standard quantum groups in a non-trivial and effective way.

The theory allows a possibility of completely pointless objects and includes, besides standard quantum groups, various geometrically interesting structures which are not quantum groups (Hopf algebras [1]), but which are more or less similar to them.

Let us start with a very simple geometrical consideration. Let G be a quantum space, represented at the formal level by a complex (unital) $*$ -algebra \mathcal{A} (consisting of appropriate “functions” on G). Geometry of G is encoded in \mathcal{A} .

As an element of the concept of a quantum space, we shall adopt a point of view according to which G naturally determines \mathcal{A} , and vice versa. In particular, spaces with non-isomorphic but Morita-equivalent “function” algebras will be interpreted as geometrically different objects (although they possess the same basic invariants like cyclic, De Rham and K-homologies [2]).

If the algebra \mathcal{A} is noncommutative then the space G can not be viewed as a structuralized collection of points. However, this does not mean that G , being a quantum object, possesses no points at all.

Actually points of G can be understood as “maps” of the form $\pi: \{\circ\} \rightarrow G$, where $\{\circ\}$ is the one-point set. Dually, at the level of function algebras, every point is determined by the corresponding “pull back” $\varphi: \mathcal{A} \rightarrow \mathbb{C}$, which is a character (nontrivial multiplicative $*$ -functional) on \mathcal{A} . Conversely, having Gelfand-Naimark theory of commutative C^* -algebras in mind (as well as a possibility of recovering points as characters in classical differential, or algebraic, geometry) it is natural to assume that every character on \mathcal{A} determines a point of G , in the way described. Of course, the space G may be “completely quantum”-without points at all.

Let us now suppose that G is endowed with a quantum group structure. By definition, the group structure on G corresponds to a Hopf algebra [1] structure on \mathcal{A} . This structure is a symbiosis of the algebra structure on \mathcal{A} , and a coalgebra

structure specified by the coproduct $\phi: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ and the counit $\epsilon: \mathcal{A} \rightarrow \mathbb{C}$ (we follow the notation of [6]). Two structures should satisfy certain additional conditions.

As first, there exists an antipodal map $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ such that

$$m(\kappa \otimes \text{id})\phi = m(\text{id} \otimes \kappa)\phi = 1\epsilon,$$

where $m: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is the multiplication in \mathcal{A} and $1 \in \mathcal{A}$ is the unit element. Secondly, the map ϕ should be multiplicative, in the sense that

$$(*) \quad \phi(ab) = \phi(a)\phi(b),$$

for each $a, b \in \mathcal{A}$. In the above relation, $\mathcal{A} \otimes \mathcal{A}$ is understood as an algebra, in a natural manner.

As a consequence of mentioned properties, it turns out that the antipode κ is an anti(co)multiplicative map. The multiplicativity of the counit is another important consequence. Further, if the group structure on G and the $*$ -structure on \mathcal{A} are mutually compatible in the sense that $\phi* = (* \otimes *)\phi$ then the composition $*\kappa$ is involutive and the counit is hermitian. In particular, the space G possesses at least one point, corresponding to the counit map (the neutral element). Let us denote by G_{cl} the “classical part” of G , consisting of all points of G . The quantum group structure on G induces, in a natural manner, a group structure on the set G_{cl} (such that G_{cl} is interpretable as a subgroup of G). As far as \mathcal{A} is noncommutative, G_{cl} is a nontrivial part of G . The space G can be imagined as a “disjoint union” of two essentially different parts: the classical part G_{cl} and the purely quantum (pointless) part $G \setminus G_{cl}$.

In this sense, quantum groups are, in contrast to ordinary ones, *inhomogeneous objects*.

The mentioned inhomogeneity is explicitly visible in the situations in which “diffeomorphisms” of G appear. All diffeomorphisms must “preserve” the classical part G_{cl} . For example, in the theory of quantum principal bundles over smooth manifolds [3] a natural correspondence between quantum G -bundles and ordinary G_{cl} -bundles (over the same manifold) holds. This is because the corresponding right-covariant “transition functions” are completely determined by their “restrictions” on G_{cl} .

On the other hand, it is natural to expect that in noncommutative geometry quantum spaces with a group structure play a similar role as Lie groups in classical differential geometry. And, among other things, Lie groups provide examples of particularly regular geometrical objects. From this point of view, the necessity of described classical-quantum decomposition of quantum groups seems strange.

Such a thinking naturally leads to an idea of generalizing a notion of a group structure on a noncommutative space, in order to include objects of a more elaborate geometrical nature. This was the main motivation for this work.

Technically speaking, the theory will be formulated in the framework of *braided algebras*. Explicitly, appropriate linear (braid) operators $\sigma: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ will enter the game. These operators play the role of the standard transposition and induce, in a natural manner, a structure of an associative algebra on $\mathcal{A} \otimes \mathcal{A}$. This requires certain compatibility conditions between σ and the algebra \mathcal{A} . We shall also

add a compatibility condition between σ and the coalgebra structure (both compatibility conditions are trivially fulfilled in the standard case). The generalization will then consist in replacing the axiom $(*)$ by a σ -relativized multiplicativity condition.

In such a way we obtain “braided quantum groups”. The attribute “braided” will be justified after an appropriate development of the formalism, by establishing that σ satisfies the braid equation.

The paper is organized as follows. The next section is devoted to the definition of braided quantum groups. In Section 3 the most important interrelations between all relevant maps will be investigated. In particular, we shall see that besides the flip-over operator σ , another braid operator $\tau: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ naturally enters the game. This operator is expressible via ϵ , ϕ and σ . Two braid operators σ and τ will play a fundamental role in the whole analysis. In particular, it will be shown that σ and τ are mutually compatible in a “braided sense”.

A theory of braided quantum groups presented in [5] can be viewed as a special case of the theory considered here.

Actually, braided quantum groups introduced in this paper reduce to the braided quantum groups of [5] iff two basic braid operators coincide. On the other hand, $\sigma = \tau$ is a necessary and sufficient condition for the multiplicativity of the counit. In particular, completely pointless braided quantum groups are not includable into the framework of [5].

A large class of examples of completely pointless braided quantum groups is given by braided Clifford algebras [4] associated to involutive braidings. This includes classical Clifford and Weyl algebras.

The paper ends with two appendices. Appendix A is devoted to the main properties of systems of braid operators, mutually compatible in a “braided sense”. A motivation for that comes from already mentioned braided compatibility between σ and τ . In particular, it will be shown that σ and τ can be naturally included in a (generally infinite) “braid system” expressing concisely all twisting properties. Finally, Appendix B discusses the question of the multiplicativity of the counit map.

2. DEFINITION OF BRAIDED QUANTUM GROUPS

Let \mathcal{A} be a complex associative algebra, with the product $m: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ and the unit element $1 \in \mathcal{A}$. Let us assume that \mathcal{A} is endowed with a coassociative coalgebra structure, specified by the coproduct $\phi: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ and the counit $\epsilon: \mathcal{A} \rightarrow \mathbb{C}$. Finally, let us assume that bijective linear maps $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ and $\sigma: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ are given such that the following equalities hold

$$\begin{aligned}
 (1) \quad & \sigma(m \otimes \text{id}) = (\text{id} \otimes m)(\sigma \otimes \text{id})(\text{id} \otimes \sigma) \\
 (2) \quad & \sigma(\text{id} \otimes m) = (m \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id}) \\
 (3) \quad & \phi m = (m \otimes m)(\text{id} \otimes \sigma \otimes \text{id})(\phi \otimes \phi) \\
 (4) \quad & (\sigma \otimes \text{id}^2)(\text{id} \otimes \phi \otimes \text{id})(\sigma^{-1} \otimes \text{id})(\text{id} \otimes \phi) \\
 & \quad \quad \quad = (\text{id}^2 \otimes \sigma)(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma^{-1})(\phi \otimes \text{id}),
 \end{aligned}$$

together with the antipode axiom

$$(5) \quad 1\epsilon = m(\text{id} \otimes \kappa)\phi = m(\kappa \otimes \text{id})\phi.$$

Definition 1. Every pair $G = (\mathcal{A}, \{\phi, \epsilon, \kappa, \sigma\})$ satisfying the above requirements is called a *braided quantum group*.

The map σ is interpretable as the “twisting operator”. In the standard theory, σ reduces to the ordinary transposition. Identities (1)–(4) express mutual compatibility between maps ϕ , m and σ . It is important to mention that the asymmetry between (1)–(2) and (4) implies that the theory is not “selfdual”. However if we replace (4) with “dual” counterparts of (1)–(2) then the theory reduces to braided quantum groups of [5] (and in particular becomes selfdual).

The space $\mathcal{A} \otimes \mathcal{A}$ is an \mathcal{A} -bimodule, in a natural manner. With the help of σ , a natural product can be defined on $\mathcal{A} \otimes \mathcal{A}$, by requiring

$$(6) \quad (a \otimes b)(c \otimes d) = a\sigma(b \otimes c)d.$$

Identities (1)–(2) ensure that this defines an associative algebra structure on $\mathcal{A} \otimes \mathcal{A}$, such that $1 \otimes 1$ is the unit element. In particular,

$$(7) \quad \sigma(1 \otimes ()) = () \otimes 1 \quad \sigma(() \otimes 1) = 1 \otimes ().$$

In the following, it will be assumed that $\mathcal{A} \otimes \mathcal{A}$ is endowed with this algebra structure. Equality (3) then says that ϕ is multiplicative.

Identity (4) expresses the coassociativity of the map $(\text{id} \otimes \sigma^{-1} \otimes \text{id})(\phi \otimes \phi)$. The “inverse” identity

$$(8) \quad (\text{id}^2 \otimes \sigma)(\text{id} \otimes \phi \otimes \text{id})(\sigma \otimes \text{id})(\text{id} \otimes \phi) \\ = (\sigma \otimes \text{id}^2)(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma)(\phi \otimes \text{id})$$

holds, too. It expresses the coassociativity of $(\text{id} \otimes \sigma \otimes \text{id})(\phi \otimes \phi)$.

3. ELEMENTARY ALGEBRAIC PROPERTIES

Let $G = (\mathcal{A}, \{\phi, \epsilon, \kappa, \sigma\})$ be a braided quantum group. As in the standard theory, the antipode is uniquely determined by (5). The flip-over operator σ is expressible through ϕ , m and κ in the following way

$$(9) \quad \sigma = (m \otimes m)(\kappa \otimes \phi m \otimes \kappa)(\phi \otimes \phi),$$

as directly follows from (3) and (5).

It is easy to see that

$$(10) \quad \phi(1) = 1 \otimes 1.$$

Indeed, $\phi(1)$ is the unity in the subalgebra $\phi(\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{A}$, as follows from (3). On the other hand, $\mathcal{A} \otimes \mathcal{A}$ is generated by $\phi(\mathcal{A})$, as a left (right) \mathcal{A} -module. Hence, $\phi(1)$ is the unity of $\mathcal{A} \otimes \mathcal{A}$. From (10) we obtain

$$(11) \quad \epsilon(1) = 1$$

$$(12) \quad \kappa(1) = 1.$$

In further computations the result of an $(n-1)$ -fold comultiplication of an element $a \in \mathcal{A}$ will be symbolically denoted by $a^{(1)} \otimes \cdots \otimes a^{(n)}$. Clearly, this element of \mathcal{A} is independent of ways in which the corresponding comultiplications are performed.

Lemma 1. *The following identities hold*

$$(13) \quad (\epsilon \otimes \text{id}) = (\text{id} \otimes \epsilon m)(\sigma \otimes \text{id})(\text{id} \otimes \phi)$$

$$(14) \quad (\text{id} \otimes \epsilon) = (\epsilon m \otimes \text{id})(\text{id} \otimes \sigma)(\phi \otimes \text{id}).$$

Proof. According to (3),

$$ab^{(1)} \otimes b^{(2)} = (\epsilon \otimes \text{id})(m \otimes m)(\text{id} \otimes \sigma \otimes \text{id})(a^{(1)} \otimes a^{(2)} \otimes b^{(1)} \otimes b^{(2)}) \otimes b^{(3)},$$

for each $a, b \in \mathcal{A}$. Acting by $m(\text{id} \otimes \kappa)$ on this equality, and using (5) we obtain

$$a\epsilon(b) = (\epsilon \otimes \text{id})(a^{(1)}\sigma(a^{(2)} \otimes b)).$$

Similarly, acting by $m(\kappa \otimes \text{id})$ on the identity

$$a^{(1)} \otimes a^{(2)}b = a^{(1)} \otimes (\text{id} \otimes \epsilon)(m \otimes m)(\text{id} \otimes \sigma \otimes \text{id})(a^{(2)} \otimes a^{(3)} \otimes b^{(1)} \otimes b^{(2)})$$

we obtain

$$\epsilon(a)b = (\text{id} \otimes \epsilon)(\sigma(a \otimes b^{(1)})b^{(2)}). \quad \square$$

A “secondary” flip-over operator τ will be now introduced in the game. From (4) we obtain

$$(15) \quad (\text{id}^2 \otimes \epsilon)(\text{id} \otimes \sigma^{-1})(\phi \otimes \text{id}) = (\epsilon \otimes \text{id}^2)(\sigma^{-1} \otimes \text{id})(\text{id} \otimes \phi).$$

Let $\tau: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ be a linear map defined by

$$(16) \quad \tau = (\text{id}^2 \otimes \epsilon)(\text{id} \otimes \sigma^{-1})(\phi \otimes \text{id})\sigma = (\epsilon \otimes \text{id}^2)(\sigma^{-1} \otimes \text{id})(\text{id} \otimes \phi)\sigma.$$

Lemma 2. *The map τ is bijective and*

$$(17) \quad \tau^{-1}\sigma = (\text{id}^2 \otimes \epsilon)(\text{id} \otimes \sigma)(\phi \otimes \text{id}) = (\epsilon \otimes \text{id}^2)(\sigma \otimes \text{id})(\text{id} \otimes \phi).$$

Proof. The second equality in (17) follows from (8). Let $\tau'\sigma$ be the map given by the second term in (17). A direct computation gives

$$\begin{aligned} \tau\tau'\sigma &= (\epsilon \otimes \text{id}^2 \otimes \epsilon)(\sigma^{-1} \otimes \text{id}^2)(\text{id} \otimes \phi \otimes \text{id})(\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\phi \otimes \text{id}) \\ &= (\epsilon \otimes \text{id}^2 \otimes \epsilon)(\sigma^{-1} \otimes \text{id}^2)(\text{id} \otimes \phi \otimes \text{id})(\sigma \otimes \epsilon \otimes \text{id})(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma)(\phi \otimes \text{id}) \\ &= (\epsilon \otimes \text{id}^2 \otimes \epsilon \otimes \epsilon)(\sigma^{-1} \otimes \text{id} \otimes \sigma)(\text{id} \otimes \phi \otimes \text{id}^2)(\text{id} \otimes \phi \otimes \text{id})(\sigma \otimes \text{id})(\text{id} \otimes \phi) \\ &= (\epsilon \otimes \text{id}^2 \otimes \epsilon \otimes \epsilon)(\text{id}^3 \otimes \sigma)(\text{id}^2 \otimes \phi \otimes \text{id})(\text{id}^2 \otimes \sigma^{-1})(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma)(\phi \otimes \text{id}) \\ &= (\text{id}^2 \otimes \epsilon \otimes \epsilon)(\text{id}^2 \otimes \sigma)(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma^{-1})(\phi \otimes \text{id})\sigma \\ &= (\text{id}^2 \otimes \epsilon \otimes \epsilon)(\sigma \otimes \text{id}^2)(\text{id} \otimes \phi \otimes \text{id})(\sigma^{-1} \otimes \text{id})(\text{id} \otimes \phi)\sigma = \sigma. \end{aligned}$$

Similarly, interchanging σ and σ^{-1} in the above computations we conclude that τ' is a left inverse for τ . Hence, τ is bijective and $\tau^{-1} = \tau'$. \square

Let us write down some important algebraic relations including the map τ . As first, let us observe that

$$(18) \quad (\epsilon \otimes \text{id})\tau = \text{id} \otimes \epsilon \quad (\text{id} \otimes \epsilon)\tau = \epsilon \otimes \text{id}$$

$$(19) \quad \tau(1 \otimes ()) = () \otimes 1 \quad \tau(() \otimes 1) = 1 \otimes ().$$

This is a direct consequence of the definition of τ , and property (7). Further, coassociativity of ϕ and relations (16)–(17) imply

$$(20) \quad (\phi \otimes \text{id})\tau^{-1}\sigma = (\text{id} \otimes \tau^{-1}\sigma)(\phi \otimes \text{id})$$

$$(21) \quad (\text{id} \otimes \phi)\tau^{-1}\sigma = (\tau^{-1}\sigma \otimes \text{id})(\text{id} \otimes \phi)$$

$$(22) \quad (\phi \otimes \text{id})\tau\sigma^{-1} = (\text{id} \otimes \tau\sigma^{-1})(\phi \otimes \text{id})$$

$$(23) \quad (\text{id} \otimes \phi)\tau\sigma^{-1} = (\tau\sigma^{-1} \otimes \text{id})(\text{id} \otimes \phi).$$

In other words, maps $\sigma\tau^{-1}$ and $\sigma^{-1}\tau$ are automorphisms of the \mathcal{A} -bicomodule $\mathcal{A} \otimes \mathcal{A}$ (with the left and the right \mathcal{A} -comodule structures given by $\phi \otimes \text{id}$ and $\text{id} \otimes \phi$ respectively). In general, $\sigma\tau^{-1}$ and $\sigma^{-1}\tau$ do not commute. However, the following commutation relations hold

$$(24) \quad (\sigma\tau^{-1} \otimes \text{id})(\text{id} \otimes \sigma\tau^{-1}) = (\text{id} \otimes \sigma\tau^{-1})(\sigma\tau^{-1} \otimes \text{id})$$

$$(25) \quad (\sigma\tau^{-1} \otimes \text{id})(\text{id} \otimes \sigma^{-1}\tau) = (\text{id} \otimes \sigma^{-1}\tau)(\sigma\tau^{-1} \otimes \text{id})$$

$$(26) \quad (\sigma^{-1}\tau \otimes \text{id})(\text{id} \otimes \sigma\tau^{-1}) = (\text{id} \otimes \sigma\tau^{-1})(\sigma^{-1}\tau \otimes \text{id})$$

$$(27) \quad (\sigma^{-1}\tau \otimes \text{id})(\text{id} \otimes \sigma^{-1}\tau) = (\text{id} \otimes \sigma^{-1}\tau)(\sigma^{-1}\tau \otimes \text{id}).$$

The above equalities follow from (20)–(23) and (16)–(17). As a direct consequence of Lemma 1 and (17) we find

$$(28) \quad \epsilon m = (\epsilon \otimes \epsilon)\sigma^{-1}\tau.$$

This generalizes the standard multiplicativity law for the counit.

Identities (4) and (8) can be rewritten in a simpler “pentagonal form”, including the operator τ and explicitly expressing twisting properties of the coproduct map.

Proposition 3. *The following identities hold*

$$(29) \quad (\phi \otimes \text{id})\sigma = (\text{id} \otimes \tau)(\sigma \otimes \text{id})(\text{id} \otimes \phi)$$

$$(30) \quad (\text{id} \otimes \phi)\sigma = (\tau \otimes \text{id})(\text{id} \otimes \sigma)(\phi \otimes \text{id})$$

$$(31) \quad (\phi \otimes \text{id})\sigma = (\text{id} \otimes \sigma)(\tau \otimes \text{id})(\text{id} \otimes \phi)$$

$$(32) \quad (\text{id} \otimes \phi)\sigma = (\sigma \otimes \text{id})(\text{id} \otimes \tau)(\phi \otimes \text{id}).$$

Proof. Using (4) and (17) we obtain

$$\begin{aligned} (\epsilon \otimes \text{id}^3)(\sigma \otimes \text{id}^2)(\text{id} \otimes \phi \otimes \text{id})(\sigma^{-1} \otimes \text{id})(\text{id} \otimes \phi) &= (\tau^{-1} \otimes \text{id})(\text{id} \otimes \phi) \\ &= (\epsilon \otimes \text{id} \otimes \sigma)(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma^{-1})(\phi \otimes \text{id}) = (\text{id} \otimes \sigma)(\phi \otimes \text{id})\sigma^{-1}. \end{aligned}$$

Similarly,

$$\begin{aligned} (\text{id}^3 \otimes \epsilon)(\text{id}^2 \otimes \sigma)(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma^{-1})(\phi \otimes \text{id}) &= (\text{id} \otimes \tau^{-1})(\phi \otimes \text{id}) \\ &= (\sigma \otimes \text{id} \otimes \epsilon)(\text{id} \otimes \phi \otimes \text{id})(\sigma^{-1} \otimes \text{id})(\text{id} \otimes \phi) = (\sigma \otimes \text{id})(\text{id} \otimes \phi)\sigma^{-1}. \end{aligned}$$

Hence, (29)–(30) hold. Starting from equalities (8) and (16) and applying the same computation we obtain (31)–(32). \square

In the next proposition “pentagonal” twisting relations including only τ are collected.

Proposition 4. *We have*

$$(33) \quad (\phi \otimes \text{id})\tau = (\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \phi)$$

$$(34) \quad (\text{id} \otimes \phi)\tau = (\tau \otimes \text{id})(\text{id} \otimes \tau)(\phi \otimes \text{id})$$

$$(35) \quad \tau(m \otimes \text{id}) = (\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes m)$$

$$(36) \quad \tau(\text{id} \otimes m) = (\tau \otimes \text{id})(\text{id} \otimes \tau)(m \otimes \text{id}).$$

Proof. Direct transformations give

$$(\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \phi) = (\text{id} \otimes \tau\sigma^{-1})(\phi \otimes \text{id})\sigma = (\phi \otimes \text{id})\tau.$$

Similarly,

$$(\tau \otimes \text{id})(\text{id} \otimes \tau)(\phi \otimes \text{id}) = (\tau\sigma^{-1} \otimes \text{id})(\text{id} \otimes \phi)\sigma = (\text{id} \otimes \phi)\tau.$$

Applying (16), (31) and (1) we obtain

$$\begin{aligned} (\text{id} \otimes m)(\tau \otimes \text{id})(\text{id} \otimes \tau) &= (\text{id} \otimes m \otimes \epsilon)(\tau \otimes \sigma^{-1})(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma) \\ &= (\text{id} \otimes m \otimes \epsilon)(\text{id}^2 \otimes \sigma^{-1})(\text{id} \otimes \sigma^{-1} \otimes \text{id})(\phi \otimes \text{id}^2)(\sigma \otimes \text{id})(\text{id} \otimes \sigma) \\ &= (\text{id}^2 \otimes \epsilon)(\text{id} \otimes \sigma^{-1})(\phi \otimes m)(\sigma \otimes \text{id})(\text{id} \otimes \sigma) \\ &= (\text{id}^2 \otimes \epsilon)(\text{id} \otimes \sigma^{-1})(\phi \otimes \text{id})\sigma(m \otimes \text{id}) = \tau(m \otimes \text{id}). \end{aligned}$$

Similarly,

$$\begin{aligned} (m \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id}) &= (\epsilon \otimes m \otimes \text{id})(\sigma^{-1} \otimes \tau)(\text{id} \otimes \phi \otimes \text{id})(\sigma \otimes \text{id}) \\ &= (\epsilon \otimes m \otimes \text{id})(\sigma^{-1} \otimes \text{id}^2)(\text{id} \otimes \sigma^{-1} \otimes \text{id})(\text{id}^2 \otimes \phi)(\text{id} \otimes \sigma)(\sigma \otimes \text{id}) \\ &= (\epsilon \otimes \text{id}^2)(\sigma^{-1} \otimes \text{id})(m \otimes \phi)(\text{id} \otimes \sigma)(\sigma \otimes \text{id}) = \tau(\text{id} \otimes m). \quad \square \end{aligned}$$

We pass to the study of algebraic relations including the antipode map. In the standard theory, the antipode is an anti(co)-multiplicative map. The next proposition gives a braided counterpart of this property.

Proposition 5. *We have*

$$(37) \quad \phi\kappa = \sigma(\kappa \otimes \kappa)\phi$$

$$(38) \quad \kappa m = m(\kappa \otimes \kappa)\tau\sigma^{-1}\tau\sigma^{-1}\tau.$$

Proof. Let us start from the identity

$$\kappa(a^{(1)})a^{(2)} \otimes a^{(3)} = 1 \otimes a.$$

Acting by $\phi \otimes \phi$ on both sides, and using (3) and (10) we obtain

$$(\phi\kappa(a^{(1)}))(a^{(2)} \otimes a^{(3)}) \otimes a^{(4)} \otimes a^{(5)} = 1 \otimes 1 \otimes a^{(1)} \otimes a^{(2)}.$$

After the action of $(\text{id} \otimes m \otimes \text{id})(\text{id}^2 \otimes \kappa \otimes \text{id})$ on both sides the above equality becomes

$$(\phi\kappa(a^{(1)}))(a^{(2)} \otimes 1) \otimes a^{(3)} = 1 \otimes \kappa(a^{(1)}) \otimes a^{(2)}.$$

Hence

$$(\phi\kappa(a^{(1)}))(a^{(2)}\kappa(a^{(3)}) \otimes 1) = (1 \otimes \kappa(a^{(1)}))(\kappa(a^{(2)}) \otimes 1).$$

Applying (6)–(7) we obtain

$$\phi\kappa(a) = \sigma(\kappa(a^{(1)}) \otimes \kappa(a^{(2)})).$$

This proves (37). To prove (38), let us start from $m(\kappa \otimes m)(\phi \otimes \text{id}) = \epsilon \otimes \text{id}$, act by it on $m \otimes m$, and apply (3) and (28). We find

$$m(\kappa \otimes m)(m \otimes m \otimes m)(\text{id} \otimes \sigma \otimes \text{id}^3)(\phi \otimes \phi \otimes \text{id}^2) = (\epsilon \otimes \epsilon)\sigma^{-1}\tau \otimes m.$$

Acting by this equality on $(\text{id}^2 \otimes \kappa \otimes \text{id})(\text{id} \otimes \phi \otimes \text{id})$ and simplifying the expression we find

$$m(\kappa m \otimes m)(\text{id} \otimes \sigma \otimes \text{id})(\phi \otimes \text{id}^2) = (\epsilon \otimes m)(\text{id} \otimes \kappa \otimes \text{id})(\sigma^{-1}\tau \otimes \text{id}).$$

Acting by this on $(\text{id}^2 \otimes \kappa)(\text{id} \otimes \sigma)(\phi \otimes \text{id})$ we obtain

$$\begin{aligned} m(\kappa m \otimes m)(\text{id} \otimes \sigma \otimes \kappa)(\text{id}^2 \otimes \sigma)(\text{id} \otimes \phi \otimes \text{id})(\phi \otimes \text{id}) \\ = (\epsilon \otimes m)(\text{id} \otimes \kappa \otimes \kappa)(\sigma^{-1}\tau \otimes \text{id})(\text{id} \otimes \sigma)(\phi \otimes \text{id}). \end{aligned}$$

After simple twisting transformations the left-hand side of the above equality becomes

$$\begin{aligned} m(\kappa m \otimes m)(\text{id}^3 \otimes \kappa)(\text{id}^2 \otimes \phi)(\text{id} \otimes \sigma\tau^{-1}\sigma)(\phi \otimes \text{id}) \\ = (\kappa m \otimes \epsilon)(\text{id} \otimes \sigma\tau^{-1}\sigma)(\phi \otimes \text{id}) = \kappa m\tau^{-1}\sigma\tau^{-1}\sigma. \end{aligned}$$

The right-hand side of the mentioned equality reduces to

$$m(\epsilon \otimes \kappa \otimes \kappa)(\sigma^{-1} \otimes \text{id})(\text{id} \otimes \phi)\sigma = m(\kappa \otimes \kappa)\tau.$$

Consequently, (38) holds. \square

Twisting properties of the antipode will be now analyzed. As first, a technical lemma

Lemma 6. *We have*

$$(39) \quad [\sigma(\kappa \otimes \text{id})\tau^{-1}\sigma\tau^{-1}(a \otimes b^{(1)})]b^{(2)} = a \otimes 1\epsilon(b)$$

$$(40) \quad a^{(1)}[\sigma(\text{id} \otimes \kappa)\tau^{-1}\sigma\tau^{-1}(a^{(2)} \otimes b)] = \epsilon(a)1 \otimes b,$$

for each $a, b \in \mathcal{A}$.

Proof. We compute

$$\begin{aligned} & (\text{id} \otimes m)(\sigma \otimes \text{id})(\kappa \otimes \text{id}^2)(\tau^{-1}\sigma\tau^{-1} \otimes \text{id})(\text{id} \otimes \phi) \\ &= (\text{id} \otimes m)(\sigma \otimes \text{id})(\kappa \otimes \text{id}^2)(\tau^{-1} \otimes \text{id})(\text{id} \otimes \phi)\sigma\tau^{-1} \\ &= (\text{id} \otimes m)(\sigma \otimes \text{id})(\kappa \otimes \sigma)(\phi \otimes \text{id})\tau^{-1} = \sigma(m \otimes \text{id})(\kappa \otimes \text{id}^2)(\phi \otimes \text{id})\tau^{-1} \\ &= \sigma(1\epsilon \otimes \text{id})\tau^{-1} = \text{id} \otimes 1\epsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} & (m \otimes \text{id})(\text{id} \otimes \sigma)(\text{id}^2 \otimes \kappa)(\text{id} \otimes \tau^{-1}\sigma\tau^{-1})(\phi \otimes \text{id}) \\ &= (m \otimes \text{id})(\text{id} \otimes \sigma)(\text{id}^2 \otimes \kappa)(\text{id} \otimes \tau^{-1})(\phi \otimes \text{id})\sigma\tau^{-1} \\ &= \sigma(\text{id} \otimes m)(\text{id} \otimes \kappa)(\text{id} \otimes \phi)\tau^{-1} = \sigma(\text{id} \otimes 1\epsilon)\tau^{-1} = 1\epsilon \otimes \text{id}. \quad \square \end{aligned}$$

Proposition 7. *The following identities hold*

$$(41) \quad \sigma(\kappa \otimes \text{id}) = (\text{id} \otimes \kappa)\tau\sigma^{-1}\tau$$

$$(42) \quad \tau(\text{id} \otimes \kappa) = (\kappa \otimes \text{id})\tau$$

$$(43) \quad \tau(\kappa \otimes \text{id}) = (\text{id} \otimes \kappa)\tau$$

$$(44) \quad \sigma(\text{id} \otimes \kappa) = (\kappa \otimes \text{id})\tau\sigma^{-1}\tau.$$

Proof. Applying Lemma 6 and property (5) we obtain

$$\sigma(\kappa \otimes \text{id})\tau^{-1}\sigma\tau^{-1}(a \otimes b) = [\sigma(\kappa \otimes \text{id})\tau^{-1}\sigma\tau^{-1}(a \otimes b^{(1)})]b^{(2)}\kappa(b^{(3)}) = a \otimes \kappa(b).$$

Similarly,

$$(\text{id} \otimes \kappa)\tau^{-1}\sigma\tau^{-1}(a \otimes b) = \kappa(a^{(1)})a^{(2)}[\sigma(\kappa \otimes \text{id})\tau^{-1}\sigma\tau^{-1}(a^{(3)} \otimes b)] = \kappa(a) \otimes b.$$

This shows (41) and (44). Using properties (16), (41), (44), (22)–(23) and (33)–(34) we obtain

$$\begin{aligned} \tau(\text{id} \otimes \kappa) &= (\epsilon \otimes \text{id}^2)(\sigma^{-1} \otimes \text{id})(\text{id} \otimes \phi)\sigma(\text{id} \otimes \kappa) \\ &= (\epsilon \otimes \kappa \otimes \text{id})(\tau^{-1}\sigma\tau^{-1} \otimes \text{id})(\text{id} \otimes \phi)\tau\sigma^{-1}\tau \\ &= (\epsilon \otimes \kappa \otimes \text{id})(\tau^{-1} \otimes \text{id})(\text{id} \otimes \phi)\tau = (\kappa \otimes \text{id})\tau. \end{aligned}$$

Similarly,

$$\begin{aligned} \tau(\kappa \otimes \text{id}) &= (\text{id}^2 \otimes \epsilon)(\text{id} \otimes \sigma^{-1})(\phi \otimes \text{id})\sigma(\kappa \otimes \text{id}) \\ &= (\text{id} \otimes \kappa \otimes \epsilon)(\text{id} \otimes \tau^{-1}\sigma\tau^{-1})(\phi \otimes \text{id})\tau\sigma^{-1}\tau \\ &= (\text{id} \otimes \kappa \otimes \epsilon)(\text{id} \otimes \tau^{-1})(\phi \otimes \text{id})\tau = (\text{id} \otimes \kappa)\tau. \quad \square \end{aligned}$$

As a direct consequence of the previous proposition we find

$$(45) \quad (\kappa \otimes \kappa)\tau = \tau(\kappa \otimes \kappa)$$

$$(46) \quad (\kappa \otimes \kappa)\sigma = \sigma(\kappa \otimes \kappa).$$

For the end of this section, we shall prove that σ and τ satisfy a system of braid equations.

Proposition 8. *The following identities hold*

$$(47) \quad (\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id}) = (\text{id} \otimes \sigma)(\sigma \otimes \text{id})(\text{id} \otimes \sigma)$$

$$(48) \quad (\tau \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id}) = (\text{id} \otimes \sigma)(\sigma \otimes \text{id})(\text{id} \otimes \tau)$$

$$(49) \quad (\sigma \otimes \text{id})(\text{id} \otimes \tau)(\sigma \otimes \text{id}) = (\text{id} \otimes \sigma)(\tau \otimes \text{id})(\text{id} \otimes \sigma)$$

$$(50) \quad (\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\tau \otimes \text{id}) = (\text{id} \otimes \tau)(\sigma \otimes \text{id})(\text{id} \otimes \sigma)$$

$$(51) \quad (\tau \otimes \text{id})(\text{id} \otimes \tau)(\sigma \otimes \text{id}) = (\text{id} \otimes \sigma)(\tau \otimes \text{id})(\text{id} \otimes \tau)$$

$$(52) \quad (\tau \otimes \text{id})(\text{id} \otimes \sigma)(\tau \otimes \text{id}) = (\text{id} \otimes \tau)(\sigma \otimes \text{id})(\text{id} \otimes \tau)$$

$$(53) \quad (\sigma \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id}) = (\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \sigma)$$

$$(54) \quad (\tau \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id}) = (\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \tau).$$

Proof. We shall first prove (48)–(51) and (53), secondly (54), thirdly (52) and finally (47). A direct computation gives

$$\begin{aligned}
(\tau \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id}) &= (\tau \otimes \text{id})(\text{id} \otimes \sigma)(m \otimes m \otimes \text{id}) \\
&\quad (\kappa \otimes \phi m \otimes \kappa \otimes \text{id})(\phi \otimes \phi \otimes \text{id}) \\
&= (\text{id} \otimes m \otimes m)(\tau \otimes \text{id}^3)(\text{id} \otimes \tau \otimes \text{id}^2)(\text{id}^2 \otimes \sigma \otimes \text{id})(\text{id}^3 \otimes \sigma) \\
&\quad (\kappa \otimes \phi m \otimes \kappa \otimes \text{id})(\phi \otimes \phi \otimes \text{id}) \\
&= (\text{id} \otimes m \otimes m)(\text{id} \otimes \kappa \otimes \phi \otimes \kappa)(\tau \otimes \text{id}^2)(\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes m \otimes \tau \sigma^{-1} \tau)(\phi \otimes \phi \otimes \text{id}) \\
&= (\text{id} \otimes m \otimes m)(\text{id} \otimes \kappa \otimes \phi m \otimes \kappa)(\tau \otimes \text{id}^3)(\text{id} \otimes \sigma \otimes \text{id}^2)(\phi \otimes \text{id} \otimes \phi)(\text{id} \otimes \tau) \\
&= (\text{id} \otimes m \otimes m)(\text{id} \otimes \kappa \otimes \phi m \otimes \kappa)(\text{id} \otimes \phi \otimes \phi)(\sigma \otimes \text{id})(\text{id} \otimes \tau) \\
&= (\text{id} \otimes \sigma)(\sigma \otimes \text{id})(\text{id} \otimes \tau).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\text{id} \otimes \tau)(\sigma \otimes \text{id})(\text{id} \otimes \sigma) &= (\text{id} \otimes \tau)(\sigma \otimes \text{id})(\text{id} \otimes m \otimes m) \\
&\quad (\text{id} \otimes \kappa \otimes \phi m \otimes \kappa)(\text{id} \otimes \phi \otimes \phi) \\
&= (m \otimes m \otimes \text{id})(\text{id}^3 \otimes \tau)(\text{id}^2 \otimes \tau \otimes \text{id})(\text{id} \otimes \sigma \otimes \text{id}^2)(\sigma \otimes \text{id}^3) \\
&\quad (\text{id} \otimes \kappa \otimes \phi m \otimes \kappa)(\text{id} \otimes \phi \otimes \phi) \\
&= (m \otimes m \otimes \text{id})(\kappa \otimes \phi \otimes \kappa \otimes \text{id})(\text{id}^2 \otimes \tau)(\text{id} \otimes \sigma \otimes \text{id})(\tau \sigma^{-1} \tau \otimes m \otimes \text{id})(\text{id} \otimes \phi \otimes \phi) \\
&= (m \otimes m \otimes \text{id})(\kappa \otimes \phi m \otimes \kappa \otimes \text{id})(\text{id}^3 \otimes \tau)(\text{id}^2 \otimes \sigma \otimes \text{id})(\phi \otimes \text{id} \otimes \phi)(\tau \otimes \text{id}) \\
&= (m \otimes m \otimes \text{id})(\kappa \otimes \phi m \otimes \kappa \otimes \text{id})(\phi \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma)(\tau \otimes \text{id}) \\
&= (\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\tau \otimes \text{id}).
\end{aligned}$$

Essentially the same transformations lead to identities (49), (51) and (53). Let us prove (54). We have

$$\begin{aligned}
(\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \tau) &= (\text{id} \otimes \tau \otimes \epsilon)(\tau \otimes \sigma^{-1})(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma) \\
&= (\text{id}^2 \otimes \epsilon \otimes \text{id})(\text{id}^2 \otimes \tau)(\text{id} \otimes \tau \otimes \text{id})(\tau \otimes \sigma^{-1})(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma) \\
&= (\text{id}^2 \otimes \epsilon \otimes \text{id})(\text{id} \otimes \sigma^{-1} \otimes \text{id})(\text{id}^2 \otimes \tau)(\text{id} \otimes \tau \otimes \text{id})(\tau \otimes \text{id}^2)(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma) \\
&= (\text{id}^2 \otimes \epsilon \otimes \text{id})(\text{id} \otimes \sigma^{-1} \otimes \text{id})(\phi \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \sigma) \\
&= (\tau \sigma^{-1} \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \sigma) = (\tau \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id}).
\end{aligned}$$

Identities (25), (48), (51) and (54) imply

$$\begin{aligned}
(\text{id} \otimes \tau)(\sigma \otimes \text{id})(\text{id} \otimes \tau) &= (\text{id} \otimes \tau)(\sigma \tau^{-1} \otimes \text{id})(\tau \otimes \text{id})(\text{id} \otimes \tau) \\
&= (\text{id} \otimes \sigma)(\sigma \tau^{-1} \otimes \text{id})(\text{id} \otimes \sigma^{-1} \tau)(\tau \otimes \text{id})(\text{id} \otimes \tau) \\
&= (\text{id} \otimes \sigma)(\sigma \tau^{-1} \otimes \text{id})(\text{id} \otimes \sigma^{-1})(\tau \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id}) \\
&= (\text{id} \otimes \sigma)(\sigma \otimes \text{id})(\text{id} \otimes \tau)(\sigma^{-1} \tau \otimes \text{id}) = (\tau \otimes \text{id})(\text{id} \otimes \sigma)(\tau \otimes \text{id}).
\end{aligned}$$

Finally, (24), (48), (50) and (52) imply

$$\begin{aligned}
(\text{id} \otimes \sigma)(\sigma \otimes \text{id})(\text{id} \otimes \sigma) &= (\text{id} \otimes \sigma\tau^{-1})(\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\tau \otimes \text{id}) \\
&= (\text{id} \otimes \sigma\tau^{-1})(\sigma\tau^{-1} \otimes \text{id})(\text{id} \otimes \tau)(\sigma \otimes \text{id})(\text{id} \otimes \tau) \\
&= (\sigma\tau^{-1} \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id})(\text{id} \otimes \tau) = (\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id}). \quad \square
\end{aligned}$$

APPENDIX A. BRAID SYSTEMS

Let us consider a complex associative algebra \mathcal{A} with the unit element $1 \in \mathcal{A}$ and the product $m: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$.

Definition 2. A *braid system* over \mathcal{A} is a collection \mathcal{F} of bijective linear maps acting in $\mathcal{A} \otimes \mathcal{A}$ and satisfying

$$(55) \quad (\alpha \otimes \text{id})(\text{id} \otimes \beta)(\gamma \otimes \text{id}) = (\text{id} \otimes \gamma)(\beta \otimes \text{id})(\text{id} \otimes \alpha)$$

$$(56) \quad \alpha(\text{id} \otimes m) = (m \otimes \text{id})(\text{id} \otimes \alpha)(\alpha \otimes \text{id})$$

$$(57) \quad \alpha(m \otimes \text{id}) = (\text{id} \otimes m)(\alpha \otimes \text{id})(\text{id} \otimes \alpha)$$

for each $\alpha, \beta, \gamma \in \mathcal{F}$.

Definition 3. A braid system \mathcal{F} is called *complete* iff it is closed under the operation $(\alpha, \beta, \gamma) \mapsto \alpha\beta^{-1}\gamma$.

Let \mathcal{F} be a braid system over \mathcal{A} . Then

$$\alpha(1 \otimes ()) = () \otimes 1 \quad \alpha(() \otimes 1) = () \otimes 1$$

for each $\alpha \in \mathcal{F}$, as follows from (56)–(57). Further, every $\alpha \in \mathcal{F}$ naturally determines an associative algebra structure on $\mathcal{A} \otimes \mathcal{A}$, with the unit element $1 \otimes 1$. The corresponding product is given by $(m \otimes m)(\text{id} \otimes \alpha \otimes \text{id})$.

We are going to prove that there exists the *minimal* complete braid system \mathcal{F}^* which extends \mathcal{F} . Starting from the system \mathcal{F} we can inductively construct an increasing chain of braid systems \mathcal{F}_n , where $n \geq 0$ and $\mathcal{F}_0 = \mathcal{F}$, while \mathcal{F}_{n+1} is consisting of maps of the form $\delta = \alpha\beta^{-1}\gamma$, where $\alpha, \beta, \gamma \in \mathcal{F}_n$. The fact that all \mathcal{F}_n are braid systems easily follows by induction, applying the definition of braid systems and the identity

$$(58) \quad (\alpha\beta^{-1} \otimes \text{id})(\text{id} \otimes \gamma\delta^{-1}) = (\text{id} \otimes \gamma\delta^{-1})(\alpha\beta^{-1} \otimes \text{id})$$

(which holds in an arbitrary braid system).

Let \mathcal{F}^* be the union of systems \mathcal{F}_n . By construction, \mathcal{F}^* is a complete braid system. Moreover, \mathcal{F}^* is the minimal braid system containing \mathcal{F} .

Let $G = (\mathcal{A}, \{\phi, \epsilon, \kappa, \sigma\})$ be a braided quantum group. According to (1)–(2), (35)–(36) and Proposition 8 operators $\{\sigma, \tau\}$ form a braid system over the algebra \mathcal{A} . The corresponding completion $\mathcal{F} = \{\sigma, \tau\}^*$ consists of maps $\sigma_n: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ of the form

$$(59) \quad \sigma_n = (\sigma\tau^{-1})^{n-1}\sigma = \sigma(\tau^{-1}\sigma)^{n-1}$$

where $n \in \mathbb{Z}$.

Proposition 9. *The following identities hold*

$$(60) \quad (\phi \otimes \text{id})\sigma_{n+k} = (\text{id} \otimes \sigma_k)(\sigma_n \otimes \text{id})(\text{id} \otimes \phi)$$

$$(61) \quad \sigma_n(\text{id} \otimes \kappa) = (\kappa \otimes \text{id})\sigma_{-n}$$

$$(62) \quad \sigma_n(\kappa \otimes \text{id}) = (\text{id} \otimes \kappa)\sigma_{-n}$$

$$(63) \quad (\text{id} \otimes \phi)\sigma_{n+k} = (\sigma_k \otimes \text{id})(\text{id} \otimes \sigma_n)(\phi \otimes \text{id}).$$

Proof. Applying Proposition 7 and (59) we obtain

$$\begin{aligned} \sigma_n(\text{id} \otimes \kappa) &= (\sigma\tau^{-1})^{n-1}\sigma(\text{id} \otimes \kappa) = (\kappa \otimes \text{id})(\tau\sigma^{-1})^{n-1}\tau\sigma^{-1}\tau \\ &= (\kappa \otimes \text{id})(\sigma\tau^{-1})^{-n-1}\sigma = (\kappa \otimes \text{id})\sigma_{-n}. \end{aligned}$$

Similarly,

$$\sigma_n(\kappa \otimes \text{id}) = (\text{id} \otimes \kappa)(\tau\sigma^{-1})^{n-1}\tau\sigma^{-1}\tau = (\text{id} \otimes \kappa)\sigma_{-n}.$$

Equalities (60) and (63) directly follow from (20)–(23) and (29)–(30). Indeed,

$$\begin{aligned} (\sigma_k \otimes \text{id})(\text{id} \otimes \sigma_n)(\phi \otimes \text{id}) &= ((\sigma\tau^{-1})^k\tau \otimes \text{id})(\text{id} \otimes \sigma(\tau^{-1}\sigma)^{n-1})(\phi \otimes \text{id}) \\ &= ((\sigma\tau^{-1})^k\tau \otimes \text{id})(\text{id} \otimes \sigma)(\phi \otimes \text{id})(\tau^{-1}\sigma)^{n-1} \\ &= ((\sigma\tau^{-1})^k \otimes \text{id})(\text{id} \otimes \phi)\sigma(\tau^{-1}\sigma)^{n-1} \\ &= (\text{id} \otimes \phi)(\sigma\tau^{-1})^{n+k-1}\sigma = (\text{id} \otimes \phi)\sigma_{n+k}. \end{aligned}$$

Similarly,

$$\begin{aligned} (\text{id} \otimes \sigma_n)(\sigma_k \otimes \text{id})(\text{id} \otimes \phi) &= (\text{id} \otimes (\sigma\tau^{-1})^n\tau)(\sigma(\tau^{-1}\sigma)^{k-1} \otimes \text{id})(\text{id} \otimes \phi) \\ &= (\phi \otimes \text{id})(\sigma\tau^{-1})^{n+k-1}\sigma = (\phi \otimes \text{id})\sigma_{n+k}. \quad \square \end{aligned}$$

As we shall now see, an arbitrary $\sigma_n \in \mathcal{F}$ is interpretable as the flip-over operator corresponding to a modified braided quantum group structure.

For each $n \in \mathbb{Z}$, let $m_n : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ and $\kappa_n : \mathcal{A} \rightarrow \mathcal{A}$ be the maps given by

$$(64) \quad m_n = m\sigma_n^{-1}\sigma$$

$$(65) \quad \kappa_n = (\epsilon \otimes \kappa)\sigma_n^{-1}\sigma\phi = (\kappa \otimes \epsilon)\sigma_n^{-1}\sigma\phi$$

(the second equality in (65) will be justified in the proof of the proposition below). It is easy to see that each m_n , interpreted as a product, determines a structure of an associative algebra on the space \mathcal{A} . Indeed,

$$\begin{aligned} m_n(m_n \otimes \text{id}) &= m\sigma_n^{-1}\sigma(m\sigma_n^{-1}\sigma \otimes \text{id}) \\ &= m\sigma_n^{-1}(\text{id} \otimes m)(\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\sigma_n^{-1}\sigma \otimes \text{id}) \\ &= m(m \otimes \text{id})(\text{id} \otimes \sigma_n^{-1})(\sigma_n^{-1}\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\sigma_n^{-1}\sigma \otimes \text{id}) \\ &= m(m \otimes \text{id})(\text{id} \otimes \sigma_n^{-1})(\sigma_n^{-1} \otimes \text{id})(\text{id} \otimes \sigma_n^{-1})(\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id}) \\ &= m(\text{id} \otimes m)(\sigma_n^{-1} \otimes \text{id})(\text{id} \otimes \sigma_n^{-1})(\sigma_n^{-1} \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id})(\text{id} \otimes \sigma) \\ &= m(\text{id} \otimes m)(\sigma_n^{-1} \otimes \text{id})(\text{id} \otimes \sigma_n^{-1}\sigma)(\sigma \otimes \text{id})(\text{id} \otimes \sigma_n^{-1}\sigma) \\ &= m\sigma_n^{-1}(m \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id})(\text{id} \otimes \sigma_n^{-1}\sigma) \\ &= m\sigma_n^{-1}\sigma(\text{id} \otimes m\sigma_n^{-1}\sigma) = m_n(\text{id} \otimes m_n). \end{aligned}$$

For each $n \in \mathbb{Z}$, let us denote by \mathcal{A}_n the vector space \mathcal{A} endowed with the product m_n . Evidently, $1 \in \mathcal{A}_n$ is the unit in this algebra, too.

Proposition 10. *The pair $G_n = (\mathcal{A}_n, \{\phi, \epsilon, \kappa_n, \sigma_n\})$ is a braided quantum group.*

Proof. We have to check the last three axioms in Definition 1. The compatibility condition between ϕ and σ_n easily follows from (60) and (63). Further, a direct computation gives

$$\begin{aligned} \phi m_n &= (m \otimes m)(\text{id} \otimes \sigma \otimes \text{id})(\phi \otimes \phi) \sigma_n^{-1} \sigma = (m \otimes m)(\text{id} \otimes \sigma \sigma_n^{-1} \sigma \otimes \text{id})(\phi \otimes \phi) \\ &= (m \otimes m)(\text{id} \otimes \sigma_{2-n} \otimes \text{id})(\phi \otimes \phi) \\ &= (m \sigma_n^{-1} \otimes m)(\text{id} \otimes \phi \otimes \text{id})(\sigma_2 \otimes \text{id})(\text{id} \otimes \phi) \\ &= (m \sigma_n^{-1} \sigma \otimes m)(\text{id} \otimes \sigma \otimes \text{id})(\phi \otimes \phi) \\ &= (m \sigma_n^{-1} \sigma \otimes m \sigma_n^{-1})(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma_{n+1})(\phi \otimes \text{id}) \\ &= (m_n \otimes m_n)(\text{id} \otimes \sigma_n \otimes \text{id})(\phi \otimes \phi). \end{aligned}$$

Finally, we have to check that k_n satisfies the antipode axiom. Let us consider maps $k_n^\pm: \mathcal{A} \rightarrow \mathcal{A}$ given by

$$k_n^- = (\kappa \otimes \epsilon) \sigma_n^{-1} \sigma \phi \quad k_n^+ = (\epsilon \otimes \kappa) \sigma_n^{-1} \sigma \phi.$$

We have

$$\begin{aligned} m_n(k_n^- \otimes \text{id})\phi &= m \sigma_n^{-1} \sigma (\epsilon \otimes \kappa \otimes \text{id})(\tau \sigma_n^{-1} \sigma \otimes \text{id})(\phi \otimes \text{id})\phi \\ &= m(\epsilon \otimes \kappa \otimes \text{id})(\text{id} \otimes \sigma_n^{-1} \sigma_{-1})(\sigma_{1-n} \otimes \text{id})(\text{id} \otimes \phi)\phi \\ &= m(\epsilon \otimes \kappa \otimes \text{id})(\text{id} \otimes \sigma_n^{-1})(\phi \otimes \text{id})\sigma_{-n}\phi \\ &= m(\epsilon \otimes \kappa \otimes \text{id})(\tau \otimes \text{id})(\text{id} \otimes \phi)\phi = m(\kappa \otimes \text{id})\phi = 1\epsilon. \end{aligned}$$

Similarly, it follows that $m_n(\text{id} \otimes \kappa_n^+)\phi = 1\epsilon$. To complete the proof, let us observe that

$$\begin{aligned} \kappa_n^+ &= (\epsilon \otimes \kappa_n^+)\phi = m_n(m_n \otimes \text{id})(\kappa_n^- \otimes \text{id} \otimes \kappa_n^+)(\phi \otimes \text{id})\phi \\ &= m_n(\text{id} \otimes m_n)(\kappa_n^- \otimes \text{id} \otimes \kappa_n^+)(\text{id} \otimes \phi)\phi = m_n(\kappa_n^- \otimes 1\epsilon)\phi = k_n^-. \end{aligned}$$

The map $k_n = k_n^\pm$ is bijective. Its inverse is given by

$$\kappa_n^{-1} \kappa = (\epsilon \otimes \text{id}) \sigma_n^{-1} \sigma_n \phi = (\text{id} \otimes \epsilon) \sigma_n^{-1} \sigma_n \phi. \quad \square$$

From the point of view of this analysis, the group G_0 is particularly interesting. For example, left-covariant first-order differential structures over G (braided counterparts of structures considered in [7]) are in a natural bijection with right \mathcal{A}_0 -ideals $\mathcal{R} \subseteq \ker(\epsilon)$. Informally speaking, G_0 is interpretable as a “maximal braided simplification” of G , with the same coalgebra structure.

APPENDIX B. A SPECIAL CASE

If G is a standard quantum group then the counit map is geometrically interpretable as a *classical point* of G . Let us analyze this question in more details, in the braided context.

Lemma 11. *The following properties are equivalent*

$$(66) \quad \epsilon m = \epsilon \otimes \epsilon$$

$$(67) \quad (\epsilon \otimes \text{id})\sigma = \text{id} \otimes \epsilon$$

$$(68) \quad (\text{id} \otimes \epsilon)\sigma = \epsilon \otimes \text{id}$$

$$(69) \quad \sigma = \tau.$$

Proof. Equality (69) implies (66), according to (28). If (66) holds then (13)–(14) imply (67)–(68). Finally, if (67) (or (68)) holds (16) implies that two flip-over operators coincide. \square

In other words, the mentioned conditions characterize the theory of [5]. The group G_0 introduced in the previous appendix is of this kind (the braiding is given by τ).

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